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SPECTRAL CONVERGENCE FOR DEGENERATING SEQUENCES OF THREE DIMENSIONAL HYPERBOLIC MANIFOLDS

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Abstract. For degenerating sequences of three dimensional hyperbolic manifolds of finite volume, we prove convergence of their eigenfunctions, heat kernel and spectral measure.

1. Introduction

A natural and important question for a degenerating sequence of Riemannian manifolds concerns their spectral behavior, i.e., spectral degeneration. The works of Wang [23] and Thurston [10] show that among locally symmetric spaces of finite volume, only hyperbolic manifolds of dimension 2 and 3 admit degenerating sequences. The spectral degeneration for hyperbolic surfaces has been intensively studied in [8] [9] [11] [12] [13] [16] [17] [24] [25] [26]. One of the results says roughly that eigenfunctions of compact surfaces converge to linear combinations of Eisenstein series of the limit non-compact surfaces [25, Theorem 3.4] [11, Theorem 1.2]. Besides its interest from the point of view of spectral convergence, this result plays an important role in [27] to verify the Phillips & Sarnak conjecture on finiteness of cuspidal spectrum for generic non-compact surfaces in special families, and is used in [15] to show that the Phillips & Sarnak conjecture implies that for a generic compact surface, almost all eigenvalues are simple. In this paper, we prove that the same result holds for degenerating sequences of three dimensional hyperbolic manifolds. We also prove the convergence of their heat kernel, Green function and spectral measure.

Let M_i, $i \geq 1$, be a degenerating sequence of three dimensional hyperbolic manifolds of finite volume converging to M_0. That is, M_i, $i \geq 1$, has finite volume and is not necessarily compact, and the lengths of several geodesics in M_i converge to zero as $i \to \infty$. Then M_0 has continuous spectrum $[1, +\infty)$, and the generalized eigenfunctions are Eisenstein series $E_{\xi_j}(u; s)$, $u \in M_0$, $s \in \mathbf{C}$, one for each end ξ_j of M_0. Our first result is the following:

Theorem 1.1. *Let φ_i be a L^2-eigenfunction of M_i with eigenvalue λ_i. Assume that $\lambda_0 = \lim_{i\to\infty} \lambda_i$ exists and $\lambda_0 \geq 1$. Then for any subsequence i', there is a further subsequence i'' such that suitable multiples of $\varphi_{i''}$ converge uniformly over compact*

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subsets to a non-zero function ψ_0 on M_0, which satisfies $\Delta\psi_0 = \lambda_0\psi_0$. Furthermore, there exist constants a_j and a L^2-eigenfunction φ_0 on M_0 of eigenvalue λ_0, which could be zero, such that

$$\psi_0(u) = \sum_j a_j E_{\xi_j}(u; s_0) + \varphi_0(u),$$

where $s_0(2 - s_0) = \lambda_0$, $\mathrm{Re}(s_0) = 1$.

Remarks. (1) If $\lambda_0 = 1$ and $E_{\xi_j}(u; 1)$ are linearly dependent, then linear combinations of derivatives $\frac{d^k}{ds^k} E_{\xi_j}(u; 1)$ should be used. (2) The same result holds if M_i is noncompact and has finite volume, and φ_i is of moderate growth instead of being square integrable (see 5.1 for the definition of moderate growth).

Our second result is the following:

Theorem 1.2. *Let $H_i(u, v, t)$ be the heat kernel of M_i, and $G_{i,s}(u, v)$, $\mathrm{Re}(s) > 2$, be the resolvent kernel of the Beltrami-Laplace operator Δ of M_i. Then $H_i(u, v, t)$ converges to $H_0(u, v, t)$ uniformly over compact subsets, and $G_{i,s}(u, v)$ converges to $G_{0,s}(u, v)$ uniformly over compact subsets away from the diagonal.*

Corollary 1.3. *The spectral measure of M_i converges to the spectral measure of M_0.*

Since there are no canonical maps between M_i and M_0, the precise definition of the convergence in the above theorems is given in 3.6. The precise statement of Corollary 1.3 is given in Proposition 6.5.

If M_i, $i \geq 1$, are assumed to be compact, Chavel and Dodziuk [3] showed that the eigenvalues of the Beltrami-Laplace operator of M_i become dense in the continuous spectrum $[1, \infty)$ of M_0 as $i \to \infty$ and they obtained the asymptotics of the rate of the spectral accumulation in terms of the lengths of the pinching geodesics. Dodziuk and McGowan [4] obtained similar results for the Hodge-Laplacian acting on differential forms. Colbois and Courtois [1] [2] showed that the eigenvalues of M_i which are less than 1 and their eigenfunctions converge to those of M_0. Theorem 1.1 generalizes the result of Colbois and Courtois and gives refined spectral convergence for the continuous spectrum. All the results in this paper hold for the Hodge-Laplacian acting on differential forms, and this will be treated elsewhere.

The organization of the rest of this paper is as follows. In §2, we recall the thick-thin decomposition of three dimensional hyperbolic manifolds. In §3, we recall deformation theory for non-compact hyperbolic manifolds and give the precise definition of the convergence of functions on M_i mentioned earlier. In §4, we derive the Maass-Selberg relation for M_i; and in §5, we prove Theorem 1.1. In §6, we prove Theorem 1.2.

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2. The Thick-Thin Decomposition

Let M be a three dimensional complete hyperbolic manifold of finite volume. For any point $u \in M$, let $\iota(u)$ be the injectivity radius at u. For any $\varepsilon > 0$, let $M_{[\varepsilon,\infty)} = \{u \in M \mid \iota(u) \geq \varepsilon\}$ be the thick part of M, and $M_{(0,\varepsilon)} = \{u \in M \mid \iota(u) < \varepsilon\}$ be the thin part of M. Then the decomposition $M = M_{[\varepsilon,\infty)} \coprod M_{(0,\varepsilon)}$ is called the thick-thin decomposition of M. According to a theorem of Kazhdan and Margulis [7, §2], there exists a universal constant $\varepsilon_0 > 0$, called Margulis constant, such that $M_{(0,\varepsilon_0)}$ consists of finitely many components, which can be described explicitly as follows.

Since M has constant curvature -1, there exists a Kleinian group $\Gamma \subset \mathrm{PSL}(2, \mathbf{C})$ such that $M = \Gamma\backslash\mathbf{H}^3$, where $\mathbf{H}^3 = \{(z,t) \mid z = x + \sqrt{-1}y \in \mathbf{C}, t > 0\}$ with the hyperbolic metric $ds^2 = t^{-2}(dx^2 + dy^2 + dt^2)$.

Assume that $\Gamma\backslash\mathbf{H}^3$ is non-compact. Then Γ contains parabolic elements. Suppose that ∞ is a parabolic fixed point of Γ, then its stabilizer Γ_∞ in Γ consists of parabolic elements of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, and hence can be identified with a lattice in $\mathbf{R}^2 \cong \mathbf{C}$.

For any $\tau > 0$, Γ_∞ preserves the horoball $\mathbf{H}^3_\tau = \{(z,t) \mid t > \tau\}$ at ∞.

Lemma 2.1. [7, §2] *There exists a positive number τ_∞ that depends only on the parabolic fixed point ∞ and Γ such that for any $\tau \geq \tau_\infty$, two points in $\mathbf{H}^3_\tau$ are Γ equivalent if and only if they are Γ_∞ equivalent. In particular, there is an embedding: $\Gamma_\infty\backslash\mathbf{H}^3_\tau \to \Gamma\backslash\mathbf{H}^3$ given by $\Gamma_\infty(z,t) \to \Gamma(z,t)$.*

The open subset $\Gamma_\infty\backslash\mathbf{H}^3_{\tau_\infty} \subset M$ is called the cusp associated with the parabolic fixed point and is denoted by C_∞. It is isometric to the space $[\tau_\infty, +\infty) \times \Gamma_\infty\backslash\mathbf{R}^2$ with the metric $t^{-2}(dt^2 + d\omega^2)$, where $d\omega^2$ is the metric on the torus $\Gamma_\infty\backslash\mathbf{R}^2$ induced from the metric $dx^2 + dy^2$ on $\mathbf{R}^2$. The constant τ_∞ satisfies the following property: For any $u \in \partial C_\infty$, the injectivity radius $\iota(u) = \varepsilon_0$. For any $a > 0$, denote the subcusp $\Gamma\backslash\mathbf{H}^3_{\tau_\infty + \exp a}$ of C_∞ by $C_\infty(a)$. Clearly, $d(\partial C_\infty, C_\infty(a)) = a$. For any other parabolic fixed point ξ of Γ, we can also define its cusp $C_\xi \subset M$ by conjugating ξ to ∞ so that for any $u \in \partial C_\xi$, $\iota(u) = \varepsilon_0$. Similarly, for any $a > 0$, we can define its subcusp $C_\xi(a) \subset C_\xi$ so that $d(\partial C_\xi, C_\xi(a)) = a$.

Any hyperbolic element $\gamma \in \Gamma$ leaves invariant a unique geodesic in $\mathbf{H}^3$ and acts on it by translation. This geodesic is called the axis of γ. For any simple closed geodesic $c(s)$ in M, its lift $\tilde{c}(s)$ in $\mathbf{H}^3$ is the axis of the deck transformation associated to γ_c. Let (r, s, θ) be the Fermi coordinates of $\mathbf{H}^3$ based on the geodesic $\tilde{c}$, where r is the distance to $\tilde{c}$, s is the arc length of $\tilde{c}$, and θ is the angular coordinate in the plane perpendicular to $\tilde{c}$ at $\tilde{c}(s)$. Then the deck transformation γ_c is given by $\gamma_c(r, s, \theta) = (r, s + |c|, \theta + \alpha)$, where $|c|$ is the length of the geodesic c in M, and $\alpha \in \mathbf{R}/2\pi$. In these coordinates, the hyperbolic metric on $\mathbf{H}^3$ is given by $ds^2 = dr^2 + \cosh^2 r ds^2 + \sinh^2 r d\theta^2$. Then for any $R > 0$, the tubular neighborhood $B(\tilde{c}, R) = \{(z,t) \in \mathbf{H}^3 \mid d((z,t), \tilde{c}) < R\}$ of $\tilde{c}$ is invariant under γ_c.

Lemma 2.2. [7, §2] *Let ε_0 be the Margulis constant. For any simple closed geodesic $c(s)$ with $|c| < 2\varepsilon_0$, there exists a constant R_c which depends only on c and M, and $R_c \sim 2\log 1/|c|$ as $|c| \to 0$ such that for any $0 < R \leq R_c$, the quotient $B(\tilde{c}, R)/\gamma_c$ embeds into M and hence is a R tubular neighborhood of c, denoted by $B(c, R)$; and for any $u \in \partial B(c, R_c)$, the injectivity radius $\iota(u) = \varepsilon_0$.*

Let $\xi_1, \cdots, \xi_p$ be the maximal set of Γ inequivalent parabolic fixed points, and $C_{\xi_1}, \cdots, C_{\xi_p}$ their cusps. Let $c_1, \cdots, c_m$ be all the simple closed geodesics in M with length less than $2\varepsilon_0$, and $R_1, \cdots, R_m$ the corresponding constants in Lemma 2.2. Then we get the following decomposition.

Lemma 2.3. [7, p. 44] *With the above notation, all the cusps $C_{\xi_1}, \cdots, C_{\xi_p}$ and the tubular neighborhoods $B(c_1, R_1), \cdots, B(c_m, R_m)$ of the simple closed geodesics*

$c_1, \ldots, c_m$ are disjoint, and

$$M_{[\varepsilon_0,\infty)} = M \setminus (\bigcup_{i=1}^{p} C_{\xi_i} \coprod \bigcup_{j=1}^{m} B(c_j, R_j)) \ , \ M_{(0,\varepsilon_0)} = \bigcup_{i=1}^{p} C_{\xi_i} \coprod \bigcup_{j=1}^{m} B(c_j, R_j).$$

In particular, there is a bijective correspondence between the ends of M and the Γ-equivalence classes of parabolic fixed points.

3. Deformation of Hyperbolic Structure

Let $M_0 = \Gamma_0 \backslash \mathbf{H}^3$ be a non-compact complete hyperbolic three dimensional manifold of finite volume. Then the deformation space of Γ_0 or M_0 in $\mathrm{PSL}(2, \mathbf{C})$ is defined by

$$\mathrm{Def}(\Gamma_0) = \mathrm{Hom}(\Gamma_0, \mathrm{PSL}(2, \mathbf{C}))/\mathrm{PSL}(2, \mathbf{C}),$$

where $\mathrm{PSL}(2, \mathbf{C})$ acts by conjugation.

Lemma 3.1. [10, Theorem 5.6] [21, Proposition 2.3] *If Γ_0 has p inequivalent parabolic fixed points, then $\dim_{\mathbf{R}} Def(\Gamma_0) = 2p$.*

Let $\overset{\wedge}{\mathbf{R}^2} = \mathbf{R}^2 \coprod \{\infty\}$ be the one point compactification of $\mathbf{R}^2$. Then we have the following parametrization of the deformation space $\mathrm{Def}(\Gamma_0)$.

Lemma 3.2. [22, p. 235] *For any choice of generators of the stabilizers $\Gamma_{0,\xi_1}, \cdots, \Gamma_{0,\xi_p}$ of all the inequivalent parabolic fixed points $\xi_1, \cdots, \xi_p$ of Γ_0, there exist a neighborhood D in $\mathrm{Def}(\Gamma_0)$ of the inclusion $i_0 : \Gamma_0 \hookrightarrow \mathrm{PSL}(2, \mathbb{C})$ and a homeomorphism from D onto a neighborhood of $(\infty, \cdots, \infty) \in (\overset{\wedge}{\mathbf{R}^2})^p$ such that i_0 is mapped to $(\infty, \cdots, \infty)$.*

For any deformation $\rho \in D \subset \mathrm{Def}(\Gamma_0)$, let $((r_1(\rho), u_1(\rho)); \cdots; (r_p(\rho), u_p(\rho))) \in (\overset{\wedge}{\mathbf{R}^2})^p$ be the corresponding coordinates, where some of $(r_i(\rho), u_i(\rho))$ could be ∞.

Lemma 3.3. [21, Theorem 5.8.2, Lemma 5.8.1] [21, Proposition 2.3] [7, p. 48] [22, Theorem 1.7, p. 235] *For any $\rho \in D \subset Def(\Gamma_0)$, the subgroup $\rho(\Gamma_0) \subset PSL(2, \mathbf{C})$ is a discrete torsion free cofinite subgroup if and only if for any $i = 1, \cdots, p$ either $r_i(\rho), u_i(\rho)$ are coprime or $(r_i(\rho), u_i(\rho)) = \infty$. For such a deformation ρ, the total number of inequivalent parabolic fixed points of $\rho(\Gamma_0)$ is equal to the number of i with $(r_i(\rho), u_i(\rho)) = \infty$.*

Definition 3.4. Assume that M_i, $i \geq 0$, is a sequence of connected complete hyperbolic manifolds of three dimensions. Then M_i is defined to converge to M_0 when $i \to \infty$ if for any $\delta > 0$ and $\varepsilon > 0$, there exist $n_0 > 0$ and a homeomorphism $f_i : M_{0,[\varepsilon,\infty)} \to M_{i,[\varepsilon,+\infty)}$ for all $i \geq n_0$ such that

$$L(f_i) = \sup_{u,v \in M_{0,[\varepsilon,\infty)}} |\log \frac{d(u, v)}{d(f_i(u), f_i(v))}| \leq \delta.$$

For any $\rho_0 \in D$ with at least one $(r_i(\rho_0), u_i(\rho_0)) = \infty$, there are only countably infinitely many ρ in a neighborhood of ρ_0 such that $\rho(\Gamma_0)$ is a torsion free cofinite subgroup, according to Lemma 3.3. Denote them by $\rho_1, \cdots, \rho_i, \cdots$. Then $\rho_i \to \rho_0$ in D as $i \to \infty$.

Proposition 3.5. [10, Theorem 5.12] *The sequence of manifolds $\rho_i(\Gamma_0)\backslash\mathbf{H}^3$ converges to $\rho_0(\Gamma_0)\backslash\mathbf{H}^3$. Conversely, if M_i is a sequence of complete hyperbolic three dimensional manifolds converging to $\rho_0(\Gamma_0)\backslash\mathbf{H}^3$, then M_i is eventually of the form $\rho_i(\Gamma_0)\backslash\mathbf{H}^3$ for some $\rho_i \in D$, and $\rho_i \to \rho_0$ in D as $i \to \infty$.*

From this proposition and the Mostow rigidity theorem, the sequence $\rho_i \in D$ with $\rho_i \neq \rho_0$ and $\rho_i \to \rho_0$ gives rise to a degenerating sequence with limit $\rho_0(\Gamma_0)\backslash\mathbf{H}^3$, since $\rho_0(\Gamma_0)\backslash\mathbf{H}^3$ is not homeomorphic to $\rho_i(\Gamma_0)\backslash\mathbf{H}^3$. By the Mostow rigidity theorem again, any convergent sequence M_i is degenerate unless it is eventually constant.

From now on, for any degenerate sequence of hyperbolic three dimensional manifolds M_i with limit M_0, we fix a uniformization group Γ_i, i.e., $M_i \cong \Gamma_i\backslash\mathbf{H}^3$, such that for $i \gg 1$, $\Gamma_i = \rho_i(\Gamma_0)$ and $\rho_i(\Gamma_0) \to \Gamma_0$ in $\mathrm{PSL}(2,\mathbf{C})$, i.e., for any $\gamma \in \Gamma_0$, $\rho_i(\gamma) \to \gamma$ as $i \to \infty$. Since there is no natural map to compare functions on M_i and M_0, we introduce the following convention.

Definition 3.6. [Convergence of Functions]. Assume that M_i converges to M_0 as $i \to \infty$. (1) Let φ_i be a continuous function on M_i for $i \geq 0$ and $\tilde{\varphi}_i$ its lift to $\mathbf{H}^3$. Then φ_i is defined to converge to φ_0 uniformly over compact subsets if $\tilde{\varphi}_i$ converges to $\tilde{\varphi}_0$ uniformly over all compact subsets of $\mathbf{H}^3$. (2) Assume that ψ_i is a function on $M_i \times M_i$ which is continuous off the diagonal, and $\tilde{\psi}_i$ is its lift on $\mathbf{H}^3 \times \mathbf{H}^3$. Then ψ_i is defined to converge to ψ_0 uniformly over compact subsets away from the diagonal if for any compact subsets $K_1, K_2 \subset \mathbf{H}^3$ with $K_1 \cap \gamma K_2 = \emptyset$ for any $\gamma \in \Gamma_0$, $\tilde{\psi}_i$ converges to $\tilde{\psi}_0$ uniformly over $K_1 \times K_2$. Similarly, we can define the uniform convergence of ψ_i to ψ_0 over all compact subsets of $\mathbf{H}^3 \times \mathbf{H}^3$.

This definition of convergence is justified by the following.

Lemma 3.7. *Assume that φ_i converges to φ_0 uniformly over compact subsets according to Definition 3.6. Then for any fixed $\varepsilon > 0$ and $f_i : M_{0,[\varepsilon,\infty)} \to M_{i,[\varepsilon,\infty)}$ as in Definition 3.4 with $L(f_i) \to 0$ as $i \to \infty$, $\varphi_i \circ f_i$ converges to φ_0 uniformly over $M_{0,[\varepsilon,\infty)}$.*

Proof. Let $\tilde{M}_{i,[\varepsilon,\infty)}$ be the inverse image in $\mathbf{H}^3$ of the thick part $M_{i,[\varepsilon,\infty)}$, and $\tilde{f}_i : \tilde{M}_{0,[\varepsilon,\infty)} \to \tilde{M}_{i,[\varepsilon,\infty)} \subset \mathbf{H}^3$ the lift of f_i. Note that $M_{0,[\varepsilon,\infty)}$ is homeomorphic to $M_{i,[\varepsilon,\infty)}$ and hence such a lift exists. Since $L(f_i) \to 0$, by the normalization of the uniformization groups Γ_i and hence the lifting, $\tilde{f}_i$ converges uniformly over compact subsets of $\tilde{M}_{i,[\varepsilon,\infty)}$ to the inclusion map $\tilde{M}_{0,[\varepsilon,\infty)} \hookrightarrow \mathbf{H}^3$. By assumption, $\tilde{\varphi}_i$ converges to $\tilde{\varphi}_0$ uniformly over compact subsets, and hence $\tilde{\varphi}_i \circ \tilde{f}_i$ converges to $\tilde{\varphi}_0$ uniformly over compact subsets. Therefore $\varphi_i \circ f_i$ converges to φ_0. □

Deformation of Γ_0 or M_0 can also be studied via deformation of fundamental domains in $\mathbf{H}^3$. For a detailed account, see [22, pp. 232-237]. Hence we recall only a few statements needed for our proof of Theorem 1.1.

Let $D_0 = D(\Gamma_0, u_0)$ be the Dirichlet domain of Γ_0 with center $u_0 \in \mathbf{H}^3$, i.e.,

$$D_0 = D(\Gamma_0, u_0) = \{u \in \mathbf{H}^3 \mid d(u, u_0) \leq d(u, \gamma u_0), \gamma \in \Gamma_0\}.$$

Then D_0 is a closed convex domain in $\mathbf{H}^3$ with finitely many faces. Points in $\overline{D_0} \cap \mathbf{H}^3(\infty)$ are parabolic fixed points of Γ_0, and any parabolic fixed point of Γ_0 is equivalent to one of them. If u_0 is generic, i.e., outside countably many totally geodesic hyperplanes in $\mathbf{H}^3$ determined by Γ_0, then any two points in $\overline{D_0} \cap \mathbf{H}^3(\infty)$

are not Γ_0 equivalent, and hence there is a one-to-one correspondence between the points in $\overline{D_0} \cap \mathbf{H}^3(\infty)$ and the cuspidal ends of M_0 [14, Lemmas 3.4 and 4.5].

For simplicity, we assume that $\overline{D_0} \cap \mathbf{H}^3(\infty) = \infty$, in particular, that M_0 has only one cusp. Then there are exactly four faces of $\overline{D_0}$ passing through ∞, and they are paired by two generators α and β of $\Gamma_{0,\infty}$. Recall from §2 that for any $\tau > 0$, $\mathbf{H}^3_\tau = \{(z,t) \in \mathbf{H}^3 \mid t > \tau\}$. Then $D_0 \cap \mathbf{H}^3_{\tau_\infty} = D(\Gamma_{0,\infty}) \times (\tau_\infty, \infty)$ (see [22, Figure 20, p. 234]), where $D(\Gamma_{0,\infty})$ is the Dirichlet fundamental domain of $\Gamma_{0,\infty}$ acting on $\mathbf{C} = \mathbf{R}^2$ with center z_0, where $u_0 = (z_0, t_0)$. The truncated domain $D_0 \setminus (D_0 \cap \mathbf{H}^3_{\tau_\infty})$ is compact.

Any deformation of Γ_0 corresponds to a deformation of D_0 in the category of convex polyhedrons. For the deformation $\Gamma_i = \rho_i(\Gamma_0)$, the compact part $D_0 \setminus (D_0 \cap \mathbf{H}^3_{\tau_\infty})$ is deformed slightly, while $D_0 \cap \mathbf{H}^3_{\tau_\infty}$ is replaced by a wedge along the common axis of helical motions $\rho_i(\alpha)$ and $\rho_i(\beta)$ (see [22, Figures 21 and 22, pp. 236-237]). Since $\rho_i(\alpha) \to \alpha$ and $\rho_i(\beta) \to \beta$, this wedge converges to $D_0 \cap \mathbf{H}^3_{\tau_\infty}$. In particular, the distance from the common axis to u_0 goes to infinity, and the dihedral angle along the wedge goes to zero as $i \to \infty$. Denote the deformed polyhedron by D_i. Then D_i is a fundamental polyhedron for Γ_i [22, Theorem 1.7, p. 235].

Since M_0 has only one cusp, M_i has exactly one pinching geodesic, denoted by c_i. Let $B(c_i, R_i)$ be the tubular neighborhood of c_i defined in Lemma 2.2. Denote the projection from $\mathbf{H}^3$ to $\Gamma_i \backslash \mathbf{H}^3$ by π_i. Then we have the following:

Lemma 3.8. *For any $a > 0$, $D_i \cap \pi_i^{-1}(B(c_i, R_i) \setminus B(c_i, R_i - a))$ converges to $D_0 \cap \pi_0^{-1}(C_\infty - C_\infty(a))$ as $i \to \infty$.*

Proof. For any $r \geq 0$, as $i \to \infty$, $D_i \setminus (D_i \cap \mathbf{H}^3_{\tau_\infty + r}) \to D_0 \setminus (D_0 \cap \mathbf{H}^3_{\tau_\infty + r})$, and hence $\iota(\pi_i(u)) \to \iota(\pi_0(u))$ for any $u \in \mathbf{H}^3$, where $\iota(\pi_i(u))$ is the injectivity radius of $\pi_i(u)$ in M_i. Since $\pi_0(D_0 \cap (\mathbf{H}^3_{\tau_\infty} \setminus \mathbf{H}^3_{\tau_\infty + e^a})) = C_\infty \setminus C_\infty(a)$ and C_∞ is defined in terms of the injectivity radius, it follows that

$$d(\pi_i(D_i \cap (\mathbf{H}^3_{\tau_\infty} \setminus \mathbf{H}^3_{\tau_\infty + e^a})), B(c_i, R_i) \setminus B(c_i, R_i - a)) \to 0,$$

and hence $d(D_i \cap \pi_i^{-1}(B(c_i, R_i) \setminus B(c_i, R_i - a)), D_i \cap (\mathbf{H}^3_{\tau_\infty} \setminus \mathbf{H}^3_{\tau_\infty + e^a})) \to 0$ as $i \to \infty$. Since $D_i \cap (\mathbf{H}^3_\tau \setminus \mathbf{H}^3_{\tau_\infty + r}) \to D_0 \cap (\mathbf{H}^3_\tau \setminus \mathbf{H}^3_{\tau_\infty + r})$, it then follows that $D_i \cap \pi_i^{-1}(B(c_i, R_i) \setminus B(c_i, R_i - a)) \to D_0 \cap \pi_0^{-1}(C_\infty \setminus C_\infty(a))$ as $i \to \infty$. □

Lemma 3.9. *For any compact subset $K \subset \mathbf{H}^3$, there exist finitely many elements $\gamma_1, \cdots, \gamma_n \in \Gamma_0$ and a positive constant a such that K is contained in $(\rho_i(\gamma_1) D_i \cup \cdots \cup \rho_i(\gamma_n) D_i) \cap \pi_1^{-1}(M_i \setminus B(c_i, R_i - a))$ for $i \gg 1$.*

Proof. Since K is compact, there exist $a > 0$ and $\gamma_1, \cdots, \gamma_n \in \Gamma_0$ such that $K \subset$ Interior$(\gamma_1 D_0 \cup \cdots \cup \gamma_n D_0 \setminus H_{\tau_\infty + e^a})$. Since $(\rho_i(\gamma_1) D_i \cup \cdots \cup \rho_i(\gamma_n) D_i) \setminus H_{\tau_\infty + e^a} \to (\gamma_1 D_0 \cup \cdots \cup \gamma_n D_0) \setminus H_{\tau_\infty + e^a}$ as $i \to \infty$, it is clear that when $i \gg 1$, $K \subset (\rho_i(\gamma_1) D_i \cup \cdots \cup \rho_i(\gamma_n) D_i) \setminus H_{\tau_\infty + e^a} = \rho_i(\gamma_1) D_i \cup \cdots \cup \rho_i(\gamma_n) D_i \setminus \pi_0^{-1}(C_\infty(a))$, and hence $K \subset (\rho_i(\gamma_1) D_i \cup \cdots \cup \rho_i(\gamma_n) D_i) \cap \pi_i^{-1}(M_i \setminus B(c_i, R_i - a))$. □

4. Maass-Selberg Relation

The Maass-Selberg relation is essential to spectral analysis for non-compact hyperbolic surfaces of finite area [18, Theorems 2.3.1 and 2.3.2]. In this section, we establish a Maass-Selberg relation for compact hyperbolic three dimensional manifolds with short geodesics. Such a formula for hyperbolic surfaces was given by

[11, Lemma 5.3]. In the previous cases, the Fourier decomposition near cusps and short geodesics plays an important role. For three dimensional manifolds, there does not exist a full Fourier decomposition near a short geodesic; instead, there exists a partial Fourier decomposition which is sufficient for our purpose here.

For simplicity, we assume that M_0 has only one end, and hence M_i has only one pinching geodesic c_i, whose length $|c_i|$ goes to zero as $i \to \infty$. Recall from Lemmas 2.2 and 2.3 that $B(c_i, R_i)$ is a tubular neighborhood of c_i homeomorphic to the interior of a solid torus, and $R_i \sim 2\log 1/|c_i|$ as $i \to \infty$. Recall further from §2 that in terms of the Fermi coordinates (r, s, θ) of $\mathbf{H}^3$, the deck transformation γ_{c_i} acts by $\gamma_{c_i}(r, s, \theta) = (r, s+|c_i|, \theta+\alpha)$. Hence r is well defined on $B(c_i, R_i)$ and measures the distance to the axial geodesic c_i; for any point $u \in B(c_i, R_i)$, this distance is denoted by $r(u)$. For any $0 < r \leq R_i$, the level surface $T_i(r) = \{u \in B(c_i, R_i) \mid r(u) = r\}$ is a torus.

Lemma 4.1. [3, §3, before Equation 3.8] *For any function φ on the tubular neighborhood $B(c_i, R_i)$, there is a partial Fourier decomposition $\varphi = \bar{\varphi} + \bar{\bar{\varphi}}$ with the following properties:*

1. $\bar{\varphi}(u) = area(T_i(r))^{-1} \int_{T_i(r)} \varphi(v)$ *and hence only depends on $r = r(u)$.*
2. $\bar{\bar{\varphi}} = \varphi - \bar{\varphi}$ *is perpendicular to constants when restricted to every torus $T_i(r)$, and hence $\bar{\bar{\varphi}}$ is perpendicular to $\bar{\varphi}$.*
3. *This decomposition is preserved by the Beltrami-Laplace operator Δ and the differentiation $\frac{d}{dr}$.*

For any $a > 0$ and a function φ on M_i, define a truncated function φ^a by cutting off the constant term $\bar{\varphi}$ on $B(c_i, R_i)$:

$$\varphi^a = \begin{cases} \bar{\bar{\varphi}} & \text{on } B(c_i, R_i - a), \\ \varphi & \text{elsewhere.} \end{cases}$$

Proposition 4.2. [Maass-Selberg Relation]. *Let φ be a smooth function on M_i satisfying $\Delta\varphi = \lambda\varphi$, $\lambda \geq 0$. Then*

$$\int_{M_i} |\bigtriangledown \varphi^a|^2 = \lambda \int_{M_i} |\varphi^a|^2 + \frac{d\,\bar{\varphi}\,(R_i - a)}{dr}\,\bar{\varphi}\; area(T_i(R_i - a)).$$

Proof. By Green's formula,

$$\int_{M_i \setminus B(c_i, R_i - a)} -\Delta\varphi\varphi + |\bigtriangledown \varphi|^2 = \int_{T_i(R_i - a)} \frac{\partial \varphi(R_i - a, w)}{\partial r} \varphi(R_i - a, w) d\mu(w),$$

where $w \in T_i(R_i - a) = \partial B(c_i, R_i - a)$, and $d\mu(w)$ is the area form on $T_i(R_i - a)$ induced from M_i. Then by Lemma 4.1,

$$\begin{aligned} \int_{M_i \setminus B(c_i, R_i - a)} -\Delta\varphi\varphi + |\bigtriangledown \varphi|^2 \quad &= \quad \int_{T_i(R_i - a)} \frac{\partial\, \bar{\bar{\varphi}}\,(R_i - a, w)}{\partial r}\, \bar{\bar{\varphi}}\,(R_i - a, w) \\ &\quad + \frac{d\,\bar{\varphi}\,(R_i - a)}{dr}\,\bar{\varphi}\,(R_i - a) d\mu(w). \end{aligned}$$

Similarly,

$$\int_{B(c_i, R_i - a)} -\Delta\, \bar{\bar{\varphi}}\bar{\bar{\varphi}} + |\bigtriangledown\, \bar{\bar{\varphi}}\,|^2 = -\int_{T_i(R_i - a)} \frac{\partial\, \bar{\bar{\varphi}}\,(R_i - a, w)}{\partial r}\, \bar{\bar{\varphi}}\,(R_i - a, w) d\mu(w).$$

Adding these two equations yields that

$$\int_{M_i} -\Delta\varphi^a\varphi^a + |\bigtriangledown \varphi^a|^2 = \frac{d\,\bar{\varphi}\,(R_i - a)}{dr}\,\bar{\varphi}\,(R_i - a)\ \mathrm{area}(T_i(R_i - a)).$$

From the assumption on φ, it follows that $\Delta\varphi^a = \lambda\varphi^a$ outside $T_i(R_i - a)$, and hence

$$-\lambda\int_{M_i} |\varphi^a|^2 + \int_{M_i} |\bigtriangledown \varphi^a|^2 = \frac{d\,\bar{\varphi}\,(R_i - a)}{dr}\,\bar{\varphi}\,(R_i - a)\ \mathrm{area}(T_i(R_i - a)).$$

□

5. Proof of Theorem 1.1

First, we need a few lemmas.

Definition 5.1. A function f on M_0 is said to have moderate growth if there exist constants $\alpha, \beta > 0$ such that

$$|f(u)| \le \alpha e^{\beta d(u,u_0)} \quad \text{for some } u_0 \in M_0, \text{ where } d(\cdot,\cdot) \text{ is the distance function.}$$

For each cusp ξ_j of M_0, an Eisenstein series $E_{\xi_j}(u; s)$, $u \in M_0$, $s \in \mathbf{C}$, is defined and has moderate growth [5]. If ∞ is a parabolic fixed point, then its Eisenstein series E_∞ is defined for $\mathrm{Re}(s) > 2$ as follows:

$$E_\infty((z,t); s) = \sum_{\gamma\in\Gamma_{0,\infty}\backslash\Gamma_0} (t_{\gamma(z,t)})^s,$$

where $t_{\gamma(z,t)}$ is the t component of the point $\gamma(z,t) \in \mathbf{H}^3$.

By a similar argument to that of [19, Satz 10 and Satz 11], we get the following result.

Lemma 5.2. *If f is a solution of $\Delta f = s(2-s)f$, $\mathrm{Re}(s) = 1$, of moderate growth, then there exist constants a_i and a L^2 solution φ_0 of $\Delta\varphi_0 = s(2-s)\varphi_0$, which is zero if $s(2-s)$ is not an eigenvalue of M_0, such that*

$$f = \sum_j a_j E_{\xi_j}(\cdot; s_0) + \varphi_0.$$

We also need the following fact.

Lemma 5.3. [3, Lemma 3.9] *For each pinching geodesic $c_{i,j}$ in M_i, let $\nu_{i,j}(r)$ be the first positive eigenvalue of the torus $T_{i,j}(r) = \partial B(c_{i,j}, r)$, then $\nu_{i,j}(r)$ is a decreasing function of r. Moreover, there exists a positive constant c independent of i such that $\nu_{i,j}(r) \ge ce^{2(R_{i,j}-r)}$ when $i \gg 1$, where $R_{i,j}$ is the width of the collar of $c_{i,j}$ in Lemma 2.2.*

We are ready to prove Theorem 1.1. Identify the function φ_i with its lift to $\mathbf{H}^3$. Then φ_i is a Γ_i invariant function on $\mathbf{H}^3$. The proof consists of three steps:

(1) Show that after suitable scaling, for any subsequence i' of i, there is a further subsequence i'' such that $\varphi_{i''}$ converges uniformly over compact subsets to a function ψ_0 on $\mathbf{H}^3$ which is invariant under Γ_0.

(2) Show that ψ_0 is not identically zero.

(3) Show that ψ_0 has moderate growth.

STEP 1

For simplicity, we assume that M_0 has only one cuspidal end, and hence M_i is compact with only one pinching geodesic c_i (see the remark at the end for the general case). We normalize φ_i by

$$\int_{M_i} |\varphi_i^a|^2 = 1,$$

where φ_i^a is the truncated function from §4. It follows from Theorem 1.1 that this normalization is essentially equivalent to two other more common normalizations:

$$\int_{M_{i,[\varepsilon_0,\infty)}} |\varphi_i|^2 = 1 \text{ , or } \sup_{M_{i,[\varepsilon_0,\infty)}} |\varphi_i| = 1.$$

However, the normalization we use here is the most convenient for the proof.

From

$$\begin{aligned}\int_{M_i} |\varphi_i^a|^2 &= \int_{M_i \setminus B(c_i,R_i-a)} |\varphi_i|^2 + \int_{B(c_i,R_i-a)} |\,\bar{\bar{\varphi}}_i\,|^2 \\ &= \int_{M_i \setminus B(c_i,R_i-a)} |\varphi_i|^2 + \int_{B(c_i,R_i)\setminus B(c_i,R_i-a)} |\,\bar{\varphi}_i\,|^2 + \int_{B(c_i,R_i)} |\,\bar{\bar{\varphi}}_i\,|^2,\end{aligned}$$

it follows that

$$\int_{B(c_i,R_i)\setminus B(c_i,R_i-a)} |\,\bar{\varphi}_i\,|^2 \le 1.$$

Since $\bar{\varphi}_i$ satisfies the following ordinary differential equation [3, Equations 2.8 and 2.9]

$$(-\frac{d^2}{dr^2} - 2\coth(2r)\frac{d}{dr})\,\bar{\varphi}_i = \lambda_i\,\bar{\varphi}_i$$

and $\coth(r) \to 1$ as $i \to \infty$ uniformly for $r \ge R_i - a$, by the stability of solutions of ordinary differential equations, we show that $\bar{\varphi}_i\,(R_i - a)$ and $\frac{d}{dr}\,\bar{\varphi}_i\,(R_i - a)$ are bounded uniformly for $i \ge 1$. By Lemma 3.8, area$(T_i(R_i - a))$ is bounded for $i \ge 1$. Then from Proposition 4.2 and the assumption $\lim_{i\to\infty} \lambda_i < +\infty$, it follows that

$$\int_{M_i} |\,\nabla \varphi_i^a|^2 \le c_1, \tag{5.1}$$

where c_1 is a constant independent of i. By the stability of ordinary differential equations again, we get that for any $b > a$,

$$\int_{B(c_i,R_i-a)\setminus B(c_i,R_i-b)} |\,\bar{\varphi}_i\,|^2 + |\frac{d}{dr}\,\bar{\varphi}_i\,|^2 \le c_2$$

for some constant c_2 independent of i. These inequalities imply that

$$\begin{aligned}\int_{M_i\setminus B(c_i,R_i-b)} |\varphi_i|^2 + |\,\nabla \varphi_i|^2 &\le \int_{M_i} |\varphi_i^a|^2 + |\,\nabla \varphi_i^a|^2 \\ &\quad + \int_{B(c_i,R_i-a)\setminus B(c_i,R_i-b)} |\,\bar{\varphi}_i\,|^2 + |\frac{d}{dr}\,\bar{\varphi}_i\,|^2 \\ &\le 1 + c_1 + c_2.\end{aligned}$$

Let $K \subset \mathbf{H}^3$ be any compact subset. Using Lemma 3.9, we get that

$$\int_K |\varphi_i|^2 + |\,\nabla \varphi_i|^2 \le c_3,$$

where c_3 is a constant independent of i. Since φ_i satisfies $\Delta\varphi_i = \lambda_i\varphi_i$ and $\lim\lambda_i$ exists, by elliptic regularity theory [6, Theorems 8.8 and 8.10], for any $k \geq 1$, there exists a constant independent of i such that

$$\|\varphi_i\|_{W^{k,2}(K)} \leq c_4,$$

where $W^{k,2}(K)$ is the Sobolev norm. Take an increasing sequence of compact subsets $K_n \subset \mathbf{H}^3$ with $\cup_{n=1}^{\infty} K_n = \mathbf{H}^3$. Then by the Sobolev embedding theorem [6, §7.7] and a diagonal argument, for any subsequence i', there is a further subsequence i'' such that $\varphi_{i''}$ converges C^k-uniformly over compact subsets to function ψ_0 on $\mathbf{H}^3$ for all $k \geq 1$. This function ψ_0 is clearly smooth and satisfies the equation $\Delta\psi_0 = \lambda_0\psi_0$, where $\lambda_0 = \lim\lambda_i$.

Since $\Gamma_i \to \Gamma_0$ as $i \to \infty$ (see the convention before Definition 3.6), and $\varphi_{i''}$ is $\Gamma_{i''}$ invariant, it follows from the uniform convergence that ψ_0 is Γ_0 invariant, and hence descends to a function on M_0.

Step 2

We show that ψ_0 is not identically zero by contradiction. If $\psi_0 \equiv 0$, then for any $b > a$, $\varphi_{i''}$ converges uniformly to zero on $M_{i''} \setminus B(c_i, R_i - b)$ as $i'' \to +\infty$. Let $\chi_{i''}$ be a cut-off function on $M_{i''}$: $\chi_{i''} = 1$ on $M_{i''} \setminus B(c_{i''}, R_{i''} - b - 1)$, $\chi_{i''} = 0$ on $B(c_{i''}, R_{i''}) \setminus B(c_{i''}, R_{i''} - b)$, $\chi_{i''}(u)$ only depends on $r(u)$, $0 \leq \chi_{i''} \leq 1$, and $|\frac{d}{dr}\chi_{i''}| \leq 2$.

Then

$$\int_{B(c_{i''}, R_{i''} - b)} |\bigtriangledown (\chi_{i''} \bar{\bar{\varphi}}_{i''})|^2 = \int_{B(c_{i''}, R_{i''} - b)} |\bigtriangledown \bar{\bar{\varphi}}_{i''}|^2 + \varepsilon_{i''}(b),$$

where

$$\varepsilon_{i''}(b) = \int_{B(c_{i''}, R_{i''} - b) \setminus B(c_{i''}, R_{i''} - b - 1)} |\bar{\bar{\varphi}}_{i''} \bigtriangledown \chi_{i''}|^2 \to 0$$

as $i'' \to \infty$ for any fixed $b > a$.

By Equation 5.1,

$$\int_{B(c_{i''}, R_{i''} - b)} |\bigtriangledown \bar{\bar{\varphi}}_{i''}|^2 \leq \int_{M_{i''}} |\bigtriangledown \varphi_{i''}^a|^2 \leq c_1,$$

where c_1 is a constant independent of i, and hence

$$\int_{B(c_{i''}, R_{i''} - b)} |\bigtriangledown (\chi_{i''} \bar{\bar{\varphi}}_{i''})|^2 \leq c_1 + |\varepsilon_{i''}(b)|. \tag{5.2}$$

On the other hand, by Lemma 5.3, there exists $b_0 > 0$ such that for any $b > b_0$, $\nu_{i''}(R_{i''} - b) \geq c_1 + 1$, and hence

$$\begin{aligned}\int_{B(c_{i''},R_{i''}-b)} |\bigtriangledown(\chi_{i''}\,\overline{\overline{\varphi}}_{i''})|^2 &\geq \int_0^{R_i-b} dr\int_{T_i(r)} |\bigtriangledown'(\chi_{i''}\,\overline{\overline{\varphi}}_{i''})|^2 \\ &\geq \int_0^{R_i-b}\nu_{i''}(r)dr\int_{T_i(r)}|\chi_{i''}\,\overline{\overline{\varphi}}_{i''}|^2 \\ &\geq \nu_{i''}(R_{i''}-b)\int_0^{R_i-b}dr\int_{T_i(r)}|\chi_{i''}\,\overline{\overline{\varphi}}_{i''}|^2 \\ &\geq (c_1+1)\int_{B(c_{i''},R_{i''}-b)}|\chi_{i''}\,\overline{\overline{\varphi}}_{i''}|^2 \\ &\geq c_1+\frac{1}{2},\end{aligned}$$

for $i'' \gg 1$, since $\int_{M_{i''}}|\varphi_{i''}|^2 = 1$ and $\varphi_{i''}$ converges uniformly to zero on $M_{i''}\setminus B(c_{i''}, R_{i''}-b)$. In the first inequality, $\bigtriangledown'$ stands for the gradient on the torus $T_i(r)$. This contradicts Inequality 5.2 when $b\to\infty$.

Step 3

We next show that ψ_0 has moderate growth on M_0. Assume that ∞ is a parabolic fixed point of Γ_0. By Lemma 3.8, for any fixed $b > a$, as $i\to+\infty$,

$$D_i\cap\pi_i^{-1}(B(c_i,R_i)\setminus B(c_i,R_i-b))\to D_0\cap\pi_0^{-1}(C_\infty\setminus C_\infty(b)),$$

where $D_i\subset \mathbf{H}^3$ is a polyhedral fundamental domain for Γ_i, and $\pi_i:\mathbf{H}^3\to\Gamma_i\backslash\mathbf{H}^3$ as in §3. From the uniform convergence of $\varphi_{i''}\to\psi_0$ over compact subsets, it then follows that the non-constant Fourier term $\overline{\overline{\varphi}}_{i''}$ on $D_{i''}\cap\pi_{i''}^{-1}(B(c_{i''},R_{i''})\setminus B(c_{i''},R_{i''}-b))$ converges uniformly to $\overline{\overline{\psi}}_0$ on $D_0\cap\pi_0^{-1}(C_\infty\setminus C_\infty(b))$, and hence

$$\begin{aligned}\int_{C_\infty\setminus C_\infty(b)}|\,\overline{\overline{\psi}}_0\,|^2 &= \int_{D_0\cap\pi_0^{-1}(C_\infty\setminus C_\infty(b))}|\,\overline{\overline{\psi}}_0\,|^2 \\ &= \lim_{i''\to\infty}\int_{D_{i''}\cap\pi_{i''}^{-1}(B(c_{i''},R_{i''})\setminus B(c_{i''},R_{i''}-b))}|\,\overline{\overline{\varphi}}_{i''}\,|^2 \\ &\leq \lim_{i\to\infty}\int_{B(c_{i''},R_{i''})}|\varphi^a_{i''}|^2 \\ &= \lim_{i\to\infty}(1-\int_{M_i\setminus B(c_{i''},R_{i''})}|\varphi^a_{i''}|^2) \\ &= 1-\int_{M_0\setminus C_\infty}|\psi_0|^2.\end{aligned}$$

Letting $b\to+\infty$, we get

$$\int_{C_\infty}|\,\overline{\overline{\psi}}_0\,|^2\leq 1. \tag{5.3}$$

Write $\lambda_0 = s_0(2-s_0)$ for some $s_0\in\mathbf{C}$ with $\mathrm{Re}(s_0)=1$. From the equation $\Delta\psi_0=\lambda_0\psi_0=s_0(2-s_0)\psi_0$, $\overline{\overline{\psi}}_0$ has the following full Fourier expansion on the cusp C_∞:

$$\overline{\overline{\psi}}_0\,(z,t)=\sum_{\tau\in\hat{\Gamma}_{0,\infty},\tau\neq 0} a_\tau tK_{s_0-1}(|\tau|t)e^{2\pi\sqrt{-1}\langle z,\tau\rangle}+b_\tau tI_{s_0-1}(|\tau|t)e^{2\pi\sqrt{-1}\langle z,\tau\rangle},$$

where $\hat{\Gamma}_{0,\infty}$ is the dual lattice of $\Gamma_{0,\infty}$ in $\mathbf{R}^2$. From the asymptotics $K_{s_0-1}(t) \sim \sqrt{\pi/2t}\exp(-t)$, $I_{s_0-1}(t) \sim 1/\sqrt{2\pi t}\exp t$ as $t \to +\infty$, and inequality (5.3) above, it is clear that $b_\tau = 0$, and hence ψ_0 has moderate growth. Then by Lemma 5.2, there exist a constant a_∞ and a L^2 solution φ_0 of $\Delta\varphi_0 = s(2-s)\varphi_0$ such that

$$\psi_0(u) = a_\infty E_\infty(u; s_0) + \varphi_0(u).$$

This completes the proof of Theorem 1.1.

Remarks. (1). In the above proof, we assume that M_0 has only one cusp and M_i are compact. If M_0 has more than one cusp and M_i are still compact, we define the truncated function φ_i^a by cutting off the constant Fourier terms at all the pinching geodesics. If M_i are non-compact, then the function φ_i^a is defined by further cutting off the constant Fourier term at every cusp of M_i; and in Step 2 we use the following counterpart of Lemma 5.3 for a cusp $C_\xi(b)$: The first positive eigenvalue of the surface $\partial C_\xi(b)$ is strictly increasing to ∞ as $b \to +\infty$.

(2). If M_i is noncompact, φ_i in Theorem 1.1 can be taken to be a linear combination of a square integrable eigenfunction and Eisenstein series of M_i, i.e., a generalized eigenfunction; and the same result holds. This condition on φ_i is equivalent to being of moderate growth, according to Lemma 5.2.

6. Proof of Theorem 1.2

We prove Theorem 1.2 in several propositions (Propositions 6.1, 6.3, 6.6). In the following, we use Definition 3.6 for the convergence of functions on M_i.

Proposition 6.1. *The heat kernel $H_i(u,v,t)$ converges to $H_0(u,v,t)$ uniformly over compact subsets in $\mathbf{H}^3 \times \mathbf{H}^3 \times (0,+\infty)$ as $i \to \infty$.*

Proof. By assumption, M_i is a degenerate sequence of hyperbolic manifolds with limit M_0. By Proposition 3.5, $M_i = \Gamma_i\backslash\mathbf{H}^3$, $M_0 = \Gamma_0\backslash\mathbf{H}^3$, and $\Gamma_i = \rho_i(\Gamma_0)$, $\rho_i(\gamma) \to \gamma$ in $\mathrm{PSL}(2,\mathbf{C})$ as $i \to \infty$ for any $\gamma \in \Gamma_0$. For any $u_0 \in \mathbf{H}^3$, there exists a small ball B_0 around u_0 of radius less than 1 such that for any $\gamma \in \Gamma_i$, $\gamma \neq 1$, $\gamma B_0 \cap B_0 = \emptyset$. This follows either from the convergence of manifolds $M_i \to M_0$ or from the convergence of the fundamental domain D_i for Γ_i to the fundamental domain D_0 for Γ_0 in §3. Fix another $v_0 \in \mathbf{H}^3$. For any $R > 0$, let $B(\Gamma_i, R) = \{\gamma \in \Gamma_i \mid d(u_0, \gamma v_0) < R\}$. Then

$$\begin{aligned}\coprod_{\gamma\in B(\Gamma_i,R)} \gamma B_0 &\subset \{v \in \mathbf{H}^3 \mid d(v,u_0) \le R + d(u_0,v_0) + 1\}\\ &= B(u_0, R + d(u_0,v_0)+1) \subset \mathbf{H}^3,\end{aligned}$$

and hence

$$|B(\Gamma_i,R)|\mathrm{vol}(B_0) \le \mathrm{vol}(B(u_0, R + d(u_0,v_0)+1)) \le c_1 e^{2R},$$

where c_1 is a constant depending only on $d(u_0,v_0)$. This implies

$$|B(\Gamma_i,R)| \le c_2 e^{2R}\ , \quad |B(\Gamma_i,R) - B(\Gamma_i,R-1)| \le c_2 e^{2R}, \tag{6.1}$$

where c_2 is a constant independent of i. By [20, p. 219], the heat kernel $P(u,v,t)$ of $\mathbf{H}^3$ is as follows:

$$P(u,v,t) = \frac{d(u,v)}{\sinh d(u,v)}(4\pi t)^{-3/2} e^{-t} e^{-d^2(u,v)/4t}.$$

Therefore we have

$$H_i(u_0,v_0,t)=\sum_{\gamma\in\Gamma_i}\frac{d(u_0,\gamma v_0)}{\sinh d(u_0,\gamma v_0)}(4\pi t)^{-3/2}e^{-t}e^{-d^2(u_0,\gamma v_0)/4t}$$
$$=\sum_{n=0}^{\infty}\sum_{\gamma\in\Gamma_i,n\le d(u_0,\gamma v_0)<n+1}\frac{d(u_0,\gamma v_0)}{\sinh d(u_0,\gamma v_0)}(4\pi t)^{-3/2}e^{-t}e^{-d^2(u_0,\gamma v_0)/4t},$$

and for any $R>0$,

$$|H_i(u_0,v_0,t)-\sum_{\gamma\in\Gamma_i,d(u_0,\gamma v_0)<R}P(u_0,\gamma v_0,t)|$$
$$\le\sum_{n\ge R}\sum_{\gamma\in\Gamma_i,n\le d(u_0,\gamma v_0)<n+1}\frac{d(u_0,\gamma v_0)}{\sinh d(u_0,\gamma v_0)}(4\pi t)^{-3/2}e^{-t}e^{-d^2(u_0,\gamma v_0)/4t}$$
$$\le\sum_{n\ge R}c_2e^{2(n+1)}\frac{n}{\sinh n}(4\pi t)^{-3/2}e^{-t}e^{-n^2/4t},$$

and hence

$$|H_i(u_0,v_0,t)-\sum_{\gamma\in\Gamma_i,d(u_0,\gamma v_0)<R}P(u_0,\gamma v_0,t)|\to 0$$

uniformly for t in compact subsets of $(0,+\infty)$.

On the other hand, from the convergence of $\Gamma_i\to\Gamma_0$, it follows that for any $R>0$ with $d(u_0,\gamma v_0)\ne R$ for all $\gamma\in\Gamma_0$,

$$\sum_{\gamma\in\Gamma_i,d(u_0,\gamma v_0)<R}P(u_0,\gamma v_0,t)\to\sum_{\gamma\in\Gamma_0,d(u_0,\gamma v_0)<R}P(u_0,\gamma v_0,t)$$

uniformly for t in compact subsets of $(0,+\infty)$. These results immediately imply the uniform convergence of $H_i(u,v,t)$ to $H_0(u,v,t)$ over compact subsets of $\mathbf{H}^3\times\mathbf{H}^3\times(0,+\infty)$. □

Proposition 6.2. *For any $t_0>0$ and any compact subsets $K_1,K_2\subset\mathbf{H}^3$ with $K_1\cap\gamma K_2=\emptyset$ for all $\gamma\in\Gamma_0$, $H_i(u,v,t)$ converges to $H_0(u,v,t)$ uniformly over $K_1\times K_2\times[0,t_0]$.*

Proof. By the proof of the previous proposition, it suffices to show that for any $\gamma\in B(\Gamma_i,R)$, $P(u,\gamma v,t)\to 0$ uniformly for $(u,v)\in K_1\times K_2$ as $t\to 0$. By assumption, $K_1\cap\gamma K_2=\emptyset$, and hence there exists $\delta>0$ such that for any $\gamma\in B(\Gamma_i,R)$, $i\ge 1$, $d(K_1,\gamma K_2)\ge\delta$, i.e., $d(u,\gamma v)\ge\delta$ for all $(u,v)\in K_1\times K_2$. Then the convergence is clear. □

Proposition 6.3. *For any $s\in\mathbf{C}$ with $\mathrm{Re}(s)>2$, the resolvent kernel $G_{i,s}(u,v)$ converges to $G_{0,s}(u,v)$ uniformly over compact subsets away from the diagonal.*

Proof. For simplicity, we assume that $s\in\mathbf{R}$, $s>2$. For any $u\ne v$,

$$G_{i,s}(u,v)=\int_0^{+\infty}e^{s(2-s)t}H_i(u,v,t)dt.$$

Since

$$H_i(u,v,t)\le H_i(u,u,t)^{\frac12}H_i(v,v,t)^{\frac12}$$

and $H_i(u,u,t)$ is decreasing in t, $H_i(u,v,t) \leq H_i(u,u,1)^{\frac{1}{2}} H_i(v,v,1)^{\frac{1}{2}}$ when $t \geq 1$. Therefore, for any $T > 1$,

$$\begin{aligned}
&|G_{i,s}(u,v) - G_{0,s}(u,v)| \\
&\qquad \leq \int_0^T e^{s(2-s)t} |H_i(u,v,t) - H_0(u,v,t)| dt \\
&\qquad\quad + \int_T^{+\infty} e^{s(2-s)t} (H_i(u,v,t) + H_0(u,v,t)) dt \\
&\qquad = \int_0^T e^{s(2-s)t} |H_i(u,v,t) - H_0(u,v,t)| dt \\
&\qquad\quad + (\sqrt{H_i(u,u,1) H_i(v,v,1)} + \sqrt{H_0(u,u,1) H_0(v,v,1)} \frac{1}{s(s-2)} e^{s(2-s)T}.
\end{aligned}$$

Since $e^{s(2-s)T} \to 0$ as $T \to +\infty$, the conclusion follows immediately from the previous proposition. □

If we use the formula for the resolvent kernel of Δ acting on $L^2(\mathbf{H}^3)$ in [20, Proposition 3.2], we can prove as in Proposition 6.1 a slightly stronger result.

Proposition 6.4. *For any $s_1, s_2 \in \mathbf{C}$ with $\mathrm{Re}(s_j) > 2$, $j = 1, 2$, then $G_{i,s_1}(u,v) - G_{i,s_2}(u,v)$ converges to $G_{0,s_1}(u,v) - G_{0,s_2}(u,v)$ uniformly over all compact subsets.*

For any $\lambda > 0$, let $F_i(\lambda; u, v)$ be the spectral measure for M_i. If M_i is compact and $\{\varphi_j\}$ is an orthonormal basis of eigenfunctions with eigenvalues $\{\lambda_j\}$, then the spectral measure $F_i(\lambda; u, v)$ of M_i is given by

$$F_i(\lambda; u, v) = \sum_{\lambda_j \leq \lambda} \varphi_j(u) \varphi_j(v).$$

If M_i is noncompact, the Eisenstein series has to be taken into account. See [8, Equation 7.2] for definition of F_i in the hyperbolic surfaces case. The same definition works for finite volume hyperbolic three dimensional manifolds.

Using $F_i(\lambda; u, v)$, Corollary 1.3 can be stated more precisely as follows:

Proposition 6.5. *For any $\lambda > 0$ which is not an eigenvalue of M_0, $F_i(\lambda; u, v)$ converges to $F_0(\lambda; u, v)$ uniformly over compact subsets as $i \to \infty$.*

The proof is the same as that of Theorem 4 in [8]. As in [8, Corollary C], this proof gives the following improvement of Proposition 6.3:

Proposition 6.6. *For $\in \mathbf{C}$, $\mathrm{Re}(s) > 1$ and $s(2-s) \notin Spec(M_0)$, $G_{i,s}(u,v)$ converges to $G_{0,s}(u,v)$ uniformly over compact subsets away from the diagonal.*

Since M_i has finite volume, $0 \in \mathrm{Spec}(M_i)$ for all $i \geq 0$, and hence $s = 2$ is a pole of $G_{i,s}(u; v)$. Let $L^2(M_i)'$ be the orthogonal complement of constants in $L^2(M_i)$. Then the Beltrami-Laplace operator Δ of M_i acts on $L^2(M_i)'$ and is invertible on this subspace. Let $G_i(u,v)$ be the Schwartz kernel of this inverse, called the Green function of M_i.

Proposition 6.7. *The Green function $G_i(u,v)$ converges to $G_0(u,v)$ uniformly over compact subsets away from the diagonal.*

Proof. Let σ be a small circle in the λ plane around $\lambda = 0$, disjoint from $\mathrm{Spec}(M_0)$. Let $s = s(\lambda)$ be the analytic branch of $s(2-s) = \lambda$ around $\lambda = 0$ with $s(0) = 2$.

Then for any $i \geq 0$,

$$G_i(u,v) = \frac{1}{2\pi\sqrt{-1}} \int_\sigma (G_{i,s(\lambda)}(u,v) - \frac{1/\mathrm{vol}(M_i)}{-\lambda}) \frac{d\lambda}{\lambda}.$$

By a theorem of Jørgensen [10, Theorem 5.12] [7, p. 46], $\mathrm{vol}(M_i) \to \mathrm{vol}(M_0)$ as $i \to \infty$. Then the conclusion follows from the previous proposition. □

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